Diffeomorphism Invariance of Geometric Descriptions of Palatini and Ashtekar Gravity *

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Abstract

In this paper, we explicitly prove the presymplectic forms of the Palatini and Ashtekar gravity to be zero along gauge orbits of the Lorentz and diffeomorphism groups, which ensures the diffeomorphism invariance of these theories.

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Geometric description and quantization [1] is the global generalization of ordinary Hamiltonian canonical description and quantization. This formulism has been shown to provide an natural way to investigate global and geometrical properties of physical systems with geometrical invariance, such as Chern-Simons theory [2], anyon system [3], and so on. But the traditional descriptions of geometrical and canonical formalism of classical

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theories are not manifestly covariant, because from the beginning one has to explicitly choose a "time" coordinate to define the canonical conjugate momenta and the initial data of systems. Several year's ago, E.Witten [4] and G.Zuckerman [5] and C.Crnkovic [6] et al. suggested a manifestly covariant geometric description, where they took the space of solutions of the classical equations as phase space. This definition is independent of any special time choice so that is manifestly covariant. Then they used this description to discuss Yang-Mills theory, string theory and general relativity etc. Recently, B.P.Dolan and K.P. Haugh [7] used Crnkovic and Witten's method to deal with the Ashtekar's gravity. They investigated the problems related to the complex nature. But a thorough discussion needs to prove the vanishing of components of presymplectic form ω tangent to the diffeomorphism and Lorentz group orbits, as Crnkovic and Witten emphasized in ref. [6] for the case of general relativity. Essentially they pointed this proof is the most delicate point in their treatment. Therefore this short paper is devoted to complete this proof for cases of Palatini and Ashtekar gravity.

The first order action of Palatini with the tetrads and Lorentz connections as its configuration space variables is given in [8]:

$$S_P(e,\omega) = \frac{1}{2}\int R_{ab}^{IJ} e_I^a e_J^b e d^4x, \tag{1}$$

where e_I^a 's are tetrads and e is the determinant of e_a^I. The curvature of the Lorentz connections ω_a^{IJ} is defined to be $R_{ab}^{IJ} = \partial_a \omega_b^{IJ} - \partial_b \omega_a^{IJ} + [\omega_a, \omega_b]^{IJ}$. Here "$a, b, c, d, ...$" stand for Riemannian indices and "$I, J, K, L, ...$" stand for the internal $SO(3,1)$ indices. The variations of the action with respect to ω_a^{IJ} and e_I^a give the equations of motion:

$$e_I^c R_{cb}^{IJ} - \frac{1}{2} R_{cd}^{MN} e_M^c e_N^d e_b^J = 0, \tag{2}$$

$$\partial_b e_I^a + \omega_{bI}{}^J e_J^a + \Gamma_{bc}^a e_I^c = 0, \tag{3}$$

in which Γ_{bc}^a is the Christoffel. The second equation of motion (3) can be written in the form:

$$\nabla_b \, e_I^a = 0, \tag{4}$$

in which $\bigtriangledown$ is torsion-free connection on both space-time and internal indices. By using equation (4), the first equation of motion (2) becomes

$$R_{ab} = 0, \tag{5}$$

which is just the Einstein field equation in vacuum.

The tangent vectors of the solution space satisfy the linearized equations of motions,

$$2 \bigtriangledown_{[c} \delta\omega_{b]}^{IJ} e_I^c = -R_{cb}^{IK} \delta(e_I^c e_{aK}) e^{aJ}, \tag{6}$$

$$\bigtriangledown_b \delta e_I^a + \delta\omega_{bI}{}^J e_J^a + \delta\Gamma_{bc}^a e_I^c = 0. \tag{7}$$

¿From the action (1), we get the presymplectic form

$$\Omega = \int_\Sigma \delta\omega_b^{IJ} \wedge \delta(e_I^a e_J^b e) d\Sigma_a, \tag{8}$$

where Σ is the space-like supersurface in space-time manifold. Obviously the presymplectic form Ω is independent of the choice of the space-like supersurface Σ and invariant under Riemannian coordinate and Lorentz transformations. But as pointed in [6], we need to prove the degeneracy along gauge orbits of Lorentz and diffeomorphism groups and, in fact, the proof is not trivial.

Under infinitesimal Lorentz transformations, the tetrads and the connections transform as

$$e_K^c \mapsto e_K^c - \alpha_K{}^J e_J^c,$$

$$\omega_b^{IJ} \mapsto \omega_b^{IJ} + \bigtriangledown_b \alpha^{IJ},$$

where α is valued in local Lorentz Lie algebra which means α will vanish at infinity, so we obtain the transformations of δe_K^c and $\delta\omega_b^{IJ}$ along the Lorentz group orbits,

$$\delta e_K^c \mapsto \delta' e_K^c = \delta e_K^c - \alpha_K{}^J e_J^c, \tag{9}$$

$$\delta\omega_b^{IJ} \mapsto \delta' \omega_b^{IJ} = \delta\omega_b^{IJ} + \bigtriangledown_b \alpha^{IJ}.$$

¿From (9) and (8), and keeping only the terms up to first order of α,

$$\Omega^{'} - \Omega = \triangle\Omega = \int_{\Sigma} \bigtriangledown_b \{\alpha^{IJ} \wedge [\delta(e^a_I e^b_J) - e^a_I e^b_J e^K_c \delta e^c_K]\} e d\Sigma_a. \tag{10}$$

where we have used equations (4), (7) and the antisymmetry of IJ in α^{IJ}. Clearly the integrand of right hand side of equation (10) is of the form $\bigtriangledown_b(X^{ab})e$ with X^{ab}, an antisymmetric tensor which makes the integral to be a surface integral at infinity. Since we restrict the local transformations in limited region, the surface integral vanishes identically. So, the presymplectic form (8) is degenerate along Lorentz group orbits.

Next, we concentrate on the proof of the degeneracy of Ω along diffeomorphism directions. The diffeomorphism transformations are in the form

$$x^a \mapsto y^a = x^a + \xi^a,$$
$$e^c_K(x) \mapsto e^b_K(y) \frac{\partial x^c}{\partial y^b}.$$

From the second equation of motion (3) and noticing $\bigtriangledown_b \xi^c = \partial_b \xi^c + \Gamma^c_{bd} \xi^d$, one obtains

$$e^c_K \mapsto e^c_K - \bigtriangledown_b \xi^c e^b_k - \omega_{dK}{}^J e^c_J \xi^d$$

and similarly,

$$\omega^{IJ}_b \mapsto \omega^{IJ}_b + \bigtriangledown_d \omega^{IJ}_b \xi^d + \omega^{IJ}_d \bigtriangledown_b \xi^d - [\omega_d, \omega_b]^{IJ} \xi^d,$$

so that

$$\delta e^c_K \mapsto \delta^{'} e^c_K = \delta e^c_K - \bigtriangledown_b \xi^c e^b_k - \omega_{dK}{}^J e^c_J \xi^d \tag{11}$$
$$\delta\omega^{IJ}_b \mapsto \delta^{'} \omega^{IJ}_b = \delta\omega^{IJ}_b + \bigtriangledown_d \omega^{IJ}_b \xi^d + \omega^{IJ}_d \bigtriangledown_b \xi^d - [\omega_d, \omega_b]^{IJ} \xi^d.$$

The presymplectic Ω can be rewritten in the form

$$\Omega = \int_{\Sigma} j^a e d\Sigma_a,$$
$$j^a = \delta\omega^{IJ}_b \wedge [-e^a_I e^b_J e^K_c \delta e^c_K + \delta e^a_I e^b_J + e^a_I \delta e^b_J]. \tag{12}$$

¿From (11) and (12) and keeping only the terms up to first order of ξ, after tedious calculations, we obtain

$$j^{'a} - j^a = \triangle j^a = \triangle j^a_1 + \triangle j^a_2 + \triangle j^a_3,$$

$$\triangle j_1^a = \nabla_d[-\delta\omega_b^{IJ} e_J^d e_I^a \wedge \xi^b + \delta\omega_b^{IJ} e_J^b \wedge (e_I^a \xi^d - e_I^d \xi^a) - \omega_b^{IJ} e_J^d e_I^a \xi^b e_c^K \wedge \delta e_K^c + \omega_b^{IJ} \xi^b \wedge (e_I^a \delta e_J^d - e_I^d \delta e_J^a)],$$

$$\triangle j_2^a = R_{bd}^{IJ}[\frac{1}{2}(e_J^b \delta e_I^d - e_I^d e^{bM} e_{cJ} \delta e_M^c) \wedge \xi^a + e_I^d e^{aM} e_{cJ} \delta e_M^c \wedge \xi^b - e_I^a e_J^b e_c^K \delta e_K^c \wedge \xi^d + \delta e_I^a e_J^b \wedge \xi^d],$$

$$\triangle j_3^a = -\delta\Gamma_{bc}^a \wedge e_J^c e_L^b \omega_d^{JL} \xi^d, \tag{13}$$

where we have used equations (3) (6) (7) and the antisymmetry of IJ in ω^{IJ}. From the deformation of the equations of motion $e_I^c R_{cb}^{IJ} = 0$, we have $\triangle j_2^a = 0$. Obviously due to the symmetry of b c in $\delta\Gamma_{bc}^a$ and the antisymmetry of JL in ${\omega_d}^{JL}$, $\triangle j_3^a = 0$. So, there only leaves with

$$\triangle\Omega = \int_\Sigma \triangle j_1^a e d\Sigma_a.$$

Like the integrand of right hand side of (10), $\triangle j_1^a$ is again of the form $\nabla_d X^{da}$ with X^{da} being an antisymmetric tensor, one gets

$$\triangle\,\Omega = \int_\Sigma \partial_d(eX^{da}) d\Sigma_a = \int_{\partial\Sigma} eX^{da} dS_{da}. \tag{14}$$

If assuming ξ^λ has compact support or, more generally, is asymptotic at infinity to a killing vector field (a more detailed discussion on boundary conditions can be found in [9]), $\triangle\Omega$ obviously vanishes which ends our proof of the degeneracy of the presymplectic form (8) of Palatini gravity along the directions of diffeomorphism transformations. If denote Z the solution space of equations of motion, G_1 the diffeomorphisms group and G_2 the Lorentz group, the presymplectic form (8) is a well defined symplectic form on the moduli space $Z/G_1/G_2$ which means that the system has constraints corresponding to diffeomorphisms transformation and local Lorentz transformation.

The same procedures of proof is suitable for Ashtekar gravity [10] [8]. Since the only difference is that in Ashtekar's case tetrads and connections are complex and self-dual (or anti-self-dual) which does not change the proof, so that we arrive at the conclusion that the diffeomorphism invariance of above geometrical description is also correct for Ashtekar gravity.

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